JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 31, No. 3, May–June 2008

Orbits and Relative Motion in the Gravitational Field of an Oblate Body

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DOI: 10.2514/1.32413

This paper studies the orbits of satellites in the noncentral gravitational field of an oblate body that includes the J_2 term. A discussion of the angular momenta and equations of motion of objects in the equatorial, near-equatorial, and polar orbits is presented. Some progress toward the solution and approximate solution of the equations of motion is presented for these cases. The paper also derives the relative-motion equations that describe a spacecraft in the vicinity of a satellite in equatorial, near-equatorial or polar orbit in this gravitational field. We show that the rendezvous equations can be simplified to resemble the corresponding equations for a central-force field. In particular, equations similar to the Tschauner–Hempel equations appear for orbits in the equatorial plane. For circular orbits in this plane, these become modified Clohessy–Wiltshire equations and possess closed-form solutions. Various types of powered optimal rendezvous problems can be formulated from these equations.

I. Introduction

ORBITS and relative motion in the gravitational field of a spherical body (or a general central-force field) have been the subject of numerous publications. We mention here only a few [1–9]. Relative-motion studies [2,4,7–9] have shown that the differential equations for the relative position of the spacecraft with respect to a satellite can be simplified considerably and, in some cases, can be solved analytically. More recently, the orbit problem [10,11] and the relative-motion problem [12] in the presence of drag (which renders the force field noncentral) have been treated by the present authors under certain simplifying assumptions.

Because of the fact that the Earth is an oblate body, its gravitational field is not exactly central. The most important correction term is the (so-called) J_2 term. With this term, the gravitational potential of the earth or other oblate spheroid is given approximately by [13,14]

$$U = -\frac{\mu}{\rho}\left[1 - \frac{R^2 J_2}{\rho^2} P_2(\cos\phi)\right] \tag{1}$$

With this equation we associate an inertial coordinate system attached to the center of the oblate body. In this system, $\boldsymbol{\rho}$ is the radius vector where $\rho = |\boldsymbol{\rho}|$, ϕ is the colatitude angle (see Fig. 1), R is the radius of the oblate body in the equatorial plane, μ represents the product of the universal gravitational constant and the mass of the spheroid, and P_2 is the second-order Legendre polynomial.

The forces due to the J_2 term as well as the quadratic drag with the upper atmosphere have important impact on the motion of clusters of satellites and recently have been the subject of considerable research efforts [10,11,15–19]. Starting with the seminal papers of Burns [14] and King-Hele and Merson [1], the impact of the J_2 term on the orbit of a particle has been treated through various forms of perturbation expansions [15–19]. Using these approaches and Lagrange's planetary equations, it was possible to obtain simple and useful expressions for the average rate of change of orbital elements. Recently a more compact approach to this problem has appeared in the literature [20]. This approach is taken in the present paper for the further study of the influence of the J_2 term on the orbits of satellites and as a vehicle for the study of relative motion under this effect.

Our first objective is to present some basic equations for angular momenta and equations of motion for various types of orbits and derive some consequences of these. Some progress is made toward the solution of the equations of motion for equatorial orbits, near-equatorial orbits, and polar orbits. This material is presented in Sec. II. An error in the differential equations that describe polar orbits in Humi [20] is corrected and new equations for polar orbits are presented.

Our second objective is to present the relative-motion equations of a spacecraft near a satellite in each of these orbits. In this analysis, the gravitational interactions between the satellite and the spacecraft are assumed to be negligible. Representative of earlier relative-motion studies in the presence of the J_2 effect are important papers by Kechichian [15], who derived a set of relative-motion equations; Gim and Alfriend [16], who completed an earlier work on a state-transition matrix; and Schweighart and Sedwick [17,18], who derived a set of approximate linear equations having closed-form solutions. The present paper presents a different approach. We express the relative-motion equations in a coordinate system moving with the satellite and show that they can be reduced to a form that closely resembles those previously derived [7,8] for a central-force field. These equations can be solved analytically if the satellite is in a circular orbit in the equatorial plane. With some additional approximations, a similar treatment can be given for the relative-motion equations between a spacecraft and a satellite in near-equatorial orbit or polar orbit. This work is found in Secs. III and IV. In Sec. V, we discuss some simulations that validate the equations derived in this paper and their accuracy. Sec. VI considers various fuel-optimal rendezvous problems associated with these new relative-motion equations for equatorial orbits and presents some necessary and sufficient conditions for their solutions. Similar formulations can be done for the other types of orbits presented. We end with a summary and conclusions in Sec. VII.

II. Satellite Orbits Around an Oblate Body

In this section we first present some general results about satellite orbits and their angular momenta. We then treat in sequence

Received 29 May 2007; revision received 20 November 2007; accepted for publication 21 November 2007.

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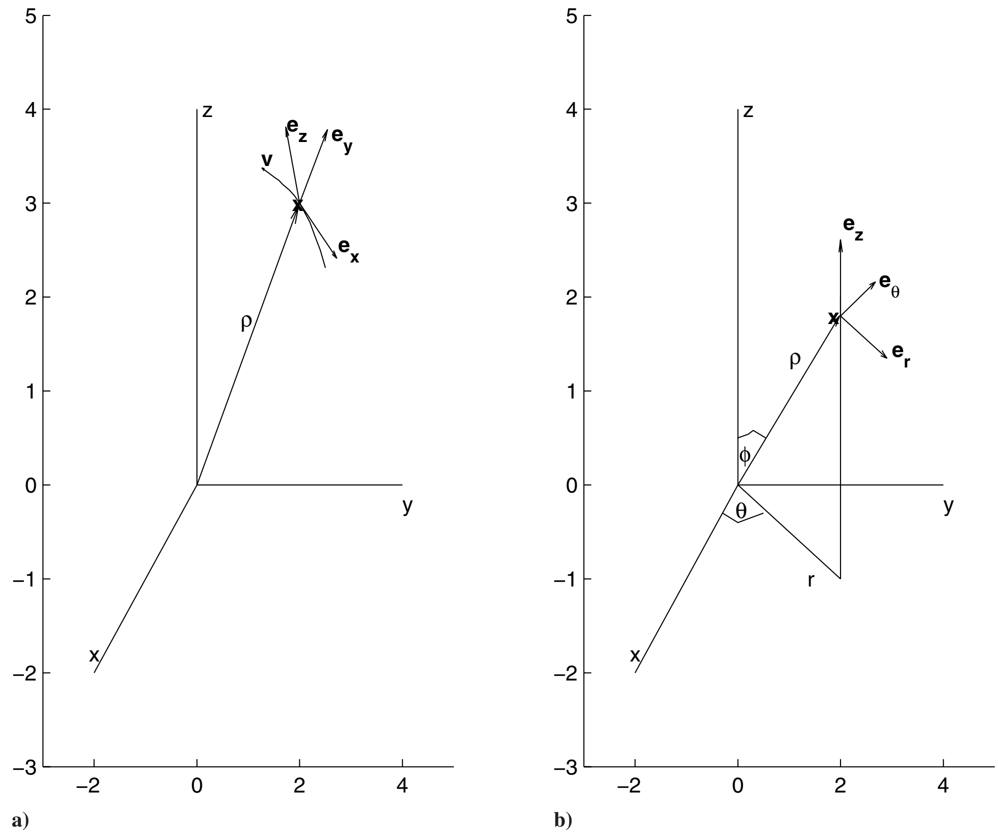

Fig. 1 The two coordinate systems used in the paper: a) the coordinate system attached to the particle (the vector v indicates the direction of the particle motion), b) the cylindrical coordinate system and the corresponding coordinate frame attached to a particle at x.

equatorial, near-equatorial, and polar orbits. We include also a discussion of circular equatorial orbits.

A. Specific Forces and Angular Momenta

The force per unit mass acting on a particle at a point $\boldsymbol{\rho} = (x, y, z)$ due to the gravitational potential (1) is given by $\mathbf{F} = -\nabla U$. It can be written as

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{2}$$

where

$$F_x = -\mu x\left[\frac{1}{\rho^3} + \frac{\frac{3}{2}R^2J_2(x^2+y^2-4z^2)}{\rho^7}\right] \tag{3}$$

$$F_y = -\mu y\left[\frac{1}{\rho^3} + \frac{\frac{3}{2}R^2J_2(x^2+y^2-4z^2)}{\rho^7}\right] \tag{4}$$

$$F_z = -\mu z\left[\frac{1}{\rho^3} + \frac{\frac{3}{2}R^2J_2(3x^2+3y^2-2z^2)}{\rho^7}\right] \tag{5}$$

We assume that the coordinates x and y are contained in the equatorial plane and that z is the polar coordinate. Using a standard cylindrical coordinate system (see Fig. 1a), it was shown [20] that the exact equations of motion of a particle in nonpolar orbit under the action of these forces are

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{L^2}\left[\frac{1}{(1+v^2)^{3/2}} + \frac{3R^2J_2u^2(1-4v^2)}{2(1+v^2)^{7/2}}\right] \tag{6}$$

$$\frac{d^2v}{d\theta^2} + v = \frac{-3\mu R^2J_2uv}{L^2(1+v^2)^{5/2}} \tag{7}$$

where $r^2 = x^2 + y^2$, $u = \frac{1}{r}$ and $v = z/r$. These equations are not valid for orbits in a polar plane because, in such a plane, θ is constant and cannot be used to parameterize the particle orbit.

It is now easy to see from Eqs. (2–5) that if the initial position and velocity of a particle are in the equatorial plane then $F_z = 0$ and the particle motion will remain in this plane. Similarly, if the initial position and velocity of a particle are in the polar plane (which without loss of generality can be taken as the x–z plane) then $F_y = 0$ and the particle motion will remain in this plane.

To treat the properties of more general orbits, we now consider the angular momentum.

By definition, the angular momentum of a particle is $\mathbf{L} = \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}$ where the dot denotes differentiation with respect to time t. To compute this expression for a particle in nonpolar orbit, we introduce a cylindrical coordinate system moving with the satellite whose unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ are given by

$$\mathbf{e}_r = (\cos\theta, \sin\theta, 0), \qquad \mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0)$$
$$\mathbf{e}_z = (0, 0, 1) \tag{8}$$

as depicted in Fig. 1. In this frame $\boldsymbol{\rho} = r\mathbf{e}_r + z\mathbf{e}_z$, hence

$$\mathbf{L} = r^2\dot{\theta}\mathbf{e}_z - rz\dot{\theta}\mathbf{e}_r - (r\dot{z} - z\dot{r})\mathbf{e}_\theta \tag{9}$$

Using the fact that $\dot{z} = z'\dot{\theta}$, $\dot{r} = r'\dot{\theta}$ where the primes denote differentiation with respect to θ, this expression becomes

$$\mathbf{L} = \dot{\theta}[r^2\mathbf{e}_z - rz\mathbf{e}_r - (rz' - zr')\mathbf{e}_\theta] \tag{10}$$

Using the same coordinate system we can derive a general expression for the rate of change of $\mathbf{L}$ for a particle in nonpolar orbit. Because $\boldsymbol{\rho} = r\mathbf{e}_r + z\mathbf{e}_z$ and $\mathbf{F} = F_r\mathbf{e}_r + F_z\mathbf{e}_z$ we have

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}) = (zF_r - rF_z)\mathbf{e}_\theta \tag{11}$$

From (3–5) we have

$$F_r = -\mu r\left[\frac{1}{(r^2+z^2)^{3/2}} + \frac{\frac{3}{2}R^2J_2(r^2-4z^2)}{(r^2+z^2)^{7/2}}\right] \tag{12}$$

$$F_z = -\mu z\left[\frac{1}{(r^2+z^2)^{3/2}} + \frac{\frac{3}{2}R^2J_2(3r^2-2z^2)}{(r^2+z^2)^{7/2}}\right] \tag{13}$$

hence

$$\frac{d\mathbf{L}}{dt} = \frac{3\mu R^2 J_2 rz}{\rho^5}\mathbf{e}_\theta \tag{14}$$

This expression can be simplified to the following:

$$\frac{d\mathbf{L}}{dt} = \frac{3\mu R^2 J_2}{2\rho^3}\sin 2\phi\mathbf{e}_\theta \tag{15}$$

We observe that this is an exact expression that is independent of any approximations.

From Eq. (14) we deduce that for a particle in the equatorial plane $\frac{d\mathbf{L}}{dt} = \mathbf{0}$ (i.e., $\mathbf{L}$ is a constant vector). From Eq. (10) we see that $\mathbf{L} = r^2\dot{\theta}\mathbf{e}_z$. For other nonpolar orbits it is easy to see that $\frac{d\mathbf{L}}{dt}\times\mathbf{L} \neq \mathbf{0}$, consequently these orbits are not planar.

Because the above treatment does not apply to particles in polar orbit, we consider this a special case now.

A polar plane is defined by a constant value of θ. However, due to the rotational symmetry around the z axis of the oblate body, we can, without loss of generality, take this plane to coincide with the x–z plane. It is natural then to parameterize the motion of a particle in a polar or near-polar orbit in terms of the angle ϕ where $0 \le \phi < 2\pi$. We define also $w^2 = x^2 + z^2$ so that $\rho^2 = w^2 + y^2$. Next we introduce the following right-handed frame that is attached to the orbiting particle:

$$\mathbf{e}_w = (\sin\phi, 0, \cos\phi), \qquad \mathbf{e}_\phi = (\cos\phi, 0, -\sin\phi) \\ \mathbf{e}_y = (0,1,0) \tag{16}$$

The reader will observe that $x = w\sin\phi$, $z = w\cos\phi$.

In this frame $\rho = w\mathbf{e}_w + y\mathbf{e}_y$. Parameterizing the orbit in terms of ϕ we then have

$$\mathbf{L} = \dot{\phi}[w^2\mathbf{e}_y + (yw' - wy')\mathbf{e}_\phi - wy\mathbf{e}_w] \tag{17}$$

where, in this context, the primes denote differentiation with respect to ϕ.

Using Eqs. (3–5) and (16) we now have

$$F_w = \mathbf{F}\cdot\mathbf{e}_w = -\frac{\mu}{2\rho^4}(2\rho^2 - 9R^2J_2\cos^2\phi + 3R^2J_2) \tag{18}$$

$$F_\phi = \mathbf{F}\cdot\mathbf{e}_\phi = \frac{3\mu}{2\rho^4}R^2J_2\sin 2\phi \tag{19}$$

Because $\frac{d\mathbf{L}}{dt} = \rho\times\mathbf{F}$, we have

$$\frac{d\mathbf{L}}{dt} = \frac{3\mu}{2\rho^4}R^2J_2\sin 2\phi\mathbf{e}_y \tag{20}$$

For a particle with initial conditions $y(0) = \frac{dy}{dt}(0) = 0$, it follows from Eq. (17) that $\mathbf{L} = w^2\dot{\phi}\mathbf{e}_y$, which implies that $\mathbf{L}$ and $\frac{d\mathbf{L}}{dt}$ are both along $\mathbf{e}_y$ at the initial time. Because Eq. (20) shows that $\frac{d\mathbf{L}}{dt}$ remains in the direction of $\mathbf{e}_y$ the motion will remain in the polar plane. It should be noted that for these orbits the length of $\mathbf{L}$ is not constant (contrary to the situation in the equatorial plane). The only exception, as seen in Eq. (20), is linear motion along a polar axis or along a line of intersection of the polar and equatorial plane. This shows that circular orbits or other Keplerian orbits cannot exist in the polar plane.

Another vector of interest in our context is the Laplace–Runge–Lenz vector

$$\mathbf{A} = \dot{\boldsymbol{\rho}}\times\mathbf{L} + U\boldsymbol{\rho} \tag{21}$$

which represents a conserved vector, namely, the eccentricity vector, for an orbit in the gravitational field of a point mass in the classical two-body problem. As can be expected from the discussion above, this vector is not conserved for an orbit in the gravitational field of an oblate body. However, one can compute the rate of change of this vector to obtain

$$\frac{d\mathbf{A}}{dt} = \mu R^2 J_2\left[\frac{1}{\rho^3}\frac{d\boldsymbol{\rho}}{dt} - \frac{3z}{\rho^5}\frac{d(z\boldsymbol{\rho})}{dt} - \frac{3z}{\rho^2}\frac{d}{dt}\left(\frac{1}{\rho}\right)\mathbf{k}\right] \tag{22}$$

B. Equatorial Orbits

If the satellite orbit is in the equatorial plane then $\boldsymbol{\rho} = (x, y, 0)$ and from Eqs. (2–5) we obtain the equation of motion:

$$\ddot{\boldsymbol{\rho}} = -\mu\left(\frac{1}{\rho^3} + \frac{3R^2J_2}{\rho^5}\right)\boldsymbol{\rho} \tag{23}$$

In this case, the satellite is in a central-force field and its motion remains always in the equatorial plane. As noted previously, its angular momentum $\mathbf{L} = \boldsymbol{\rho}\times\dot{\boldsymbol{\rho}}$ is constant and is always along the z axis. Moreover, $\rho = r$, and if we introduce polar coordinates (r, θ) in the equatorial plane we have

$$r^2\dot{\theta} = L \tag{24}$$

where $L = |\mathbf{L}|$. Letting $u(\theta) = \frac{1}{r(\theta)}$ and following the standard derivation for the equation of motion in a central-force field [21], we obtain the following equation for the satellite orbit:

$$u'' + u = \frac{\mu}{L^2}\left(1 + \frac{3}{2}R^2J_2u^2\right) \tag{25}$$

where here the primes denote differentiation with respect to θ. We observe that this equation has solutions where u is constant, demonstrating that circular orbits are allowed.

1. *First Integral*

Introducing the constants,

$$k_1 = \frac{\mu}{L^{3/2}}, \qquad k_2 = \frac{3\mu R^2 J_2}{2L^{5/2}} \tag{26}$$

we can rewrite Eq. (25) in the form

$$-\frac{u''}{u} + \frac{k_1}{L^{1/2}u} + k_2L^{1/2}u = 1 \tag{27}$$

This form of the equation of motion will be useful in the following sections.

Equation (25) can be solved implicitly by quadratures as follows. We multiply by u' and integrate with respect to θ to obtain an integral implicit in the following expression:

$$(u')^2 + u^2 = \frac{\mu}{L^2}(2u + R^2J_2u^3) + C_1 \tag{28}$$

where C_1 is an integration constant. Solving this equation for u' leads to

$$\theta + C_2 = \pm\int^u \frac{du}{\sqrt{\frac{\mu}{L^2}(2u + R^2J_2u^3) - u^2 + C_1}} \tag{29}$$

where C_2 is a second constant of integration. This indefinite integral can be expressed in terms of elliptic functions [22].

When $\rho \gg R$, the J_2 term becomes negligible and the solution can be approximated by the standard Keplerian motion:

$$u = A(1 + \epsilon\cos(\theta - \theta_0)) \tag{30}$$

where A, ϵ, θ_0 are well-known constants. Respectively, A is the reciprocal of the semilatus rectum, ϵ is the eccentricity, and θ_0 is the argument of the perigee.

2. Circular Equatorial Orbits

We have established that Keplerian orbits having constant angular momentum cannot exist outside the equatorial plane. We demonstrate now that the only Keplerian orbits that can exist in this plane are circular.

Any Keplerian orbit can be expressed in the form (30) where $\epsilon \geq 0$. Substituting this expression in Eq. (25), we observe that the equation is satisfied if and only if $\epsilon = 0$. In this case, $u = u_0 = \frac{1}{r_0}$ is a constant. The orbit is circular of radius r_0 and

$$2r_0^2 - \frac{2L^2}{\mu} r_0 + 3R^2 J_2 = 0 \tag{31}$$

Initially a reader might think that for a given angular momentum L there could be two circular orbits defined by Eq. (31). We shall show that this is not the case if $J_2 \leq \frac{2}{3}$.

Solving Eq. (31) for L we find

$$L = \pm\sqrt{\mu}\left(r_0 + \frac{3R^2 J_2}{2r_0}\right)^{1/2} \tag{32}$$

This shows that given $r_0 > R$ an orbit can be clockwise or counterclockwise, but within this restriction there is only one angular momentum L and one angular velocity $\omega = L/r_0^2$. Henceforth, we shall consider only the positive sign associated with Eq. (32). Considering L as a function of r_0 in Eq. (32), it is evident that L has a minimum at $r_0 = R\sqrt{\frac{3}{2}J_2}$ and is increasing for $r_0 > R\sqrt{\frac{3}{2}J_2}$. It is necessary that $r_0 > R$, therefore, if $J_2 \leq 2/3$ then Eq. (31) can have only one root where $r_0 > R$. Also, because L is greater at r_0 than at R, we must have $L > \sqrt{\mu R(1 + \frac{3}{2}J_2)}$. Conversely, if this condition holds, $J_2 \leq 2/3$ and r_0 is associated with an admissible root of Eq. (31), then it is the radius of a unique equatorial circular orbit having angular velocity $\omega_0 = L/r_0^2$ and period $t_p = 2\pi/\omega_0$. These results can be summarized as follows:

Theorem: Given an angular momentum

$$L > \sqrt{\mu R\left(1 + \frac{3}{2}J_2\right)} \tag{33}$$

then a unique circular orbit having a radius $r_0 > R$ exists in the equatorial plane if and only if

$$J_2 \leq 2/3 \tag{34}$$

This radius is defined by the formula

$$r_0 = \frac{L^2}{2\mu}\left[1 + \sqrt{1 - \frac{6\mu^2 R^2 J_2}{L^4}}\right] \tag{35}$$

the angular velocity by

$$\omega_0 = \frac{L}{r_0^2} \tag{36}$$

and the orbital period by

$$t_p = \frac{2\pi r_0^2}{L} \tag{37}$$

Remark: In order for two circular orbits to have the same angular momenta, the inequality (34) must be violated. There is no known planet or moon in the solar system that is this oblate.

C. Near-Equatorial Orbits

The exact equations of motion [20] of a satellite in a general nonpolar orbit are

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{\mu}{L^2}\left[\frac{1}{(1+v^2)^{3/2}} + \frac{3R^2 J_2 u^2(1-4v^2)}{2(1+v^2)^{7/2}}\right] \tag{38}$$

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\theta^2} + v = \frac{-3\mu R^2 J_2 u v}{L^2(1+v^2)^{5/2}} \tag{39}$$

where $u = \frac{1}{r}$ and $v = z/r$. If the satellite is in a near-equatorial orbit, then $v \ll 1$ and we can approximate Eq. (38) by Eq. (25); Eq. (39) reduces to

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\theta^2} + \left(1 + \frac{3\mu R^2 J_2 u}{L^2}\right) v = 0 \tag{40}$$

With this approximation the solution for u is known in terms of elliptic functions, hence Eq. (40) is a linear equation in v. In cases where u can be approximated by Eq. (30), then the general solution for v can be expressed in terms of Mathieu functions [22]. An explicit solution for v in terms of elementary functions can be obtained if we can neglect completely the J_2 term in Eq. (40) (i.e., we assume that $R^2 u \ll 1$). This leads to

$$v = B\cos\nu(\theta - \theta_1) \tag{41}$$

where $\nu^2 = 1 + \frac{3\mu R^2 J_2 u}{L^2}$. In this case we have the approximation $z = Br\cos\nu(\theta - \theta_1)$.

To treat the angular momentum under the approximation $v \ll 1$, we note that the size of the components of $\mathbf{L}$ along $\mathbf{e}_r$ and $\mathbf{e}_\theta$ relative to the component along $\mathbf{e}_z$ is

$$\frac{L_r}{L_z} = v, \qquad \frac{L_\theta}{L_z} = \frac{z'}{r} - \frac{zr'}{r^2} = \frac{z'}{r} + zu' \tag{42}$$

Results of simulations (as presented in Fig. 6) indicate that for near-equatorial orbits, $\frac{L_\theta}{L_z} \ll 1$. Consequently, for satellites in near-equatorial orbits where $|v| \ll 1$ we approximate the angular momentum vector by its z component (i.e., $\mathbf{L} = r^2\dot{\theta}\mathbf{k}$), which is constant [20]

D. Polar Orbits

To derive the equations of motion for a particle in polar orbit, we use the coordinate frame attached to the satellite that was defined in Eq. (16). In this frame we have

$$\ddot{w} - w\dot{\phi}^2 = -\frac{\mu}{2\rho^4}(2\rho^2 - 9R^2 J_2 \cos^2\phi + 3R^2 J_2) \tag{43}$$

$$w\ddot{\phi} + 2\dot{w}\dot{\phi}^2 = \frac{3\mu}{2w^4} R^2 J_2 \sin 2\phi \tag{44}$$

Multiplying Eq. (44) by w and integrating, we obtain

$$L = w^2\dot{\phi} = h + \frac{3\mu}{2} R^2 J_2 \int \frac{\sin 2\phi}{w^3}\,\mathrm{d}t \tag{45}$$

where again $L = |\mathbf{L}|$ and h is a constant of integration. However, another relation can be obtained if we multiply Eq. (44) by $w^2\dot{\phi}$ and integrate:

$$L^2 = H + 3\mu R^2 J_2 \int \frac{\sin 2\phi}{w}\,\mathrm{d}\phi \tag{46}$$

By setting $J_2 = 0$, we note that $H = h^2$. It now follows from Eq. (45) that

$$\dot{\phi} = \frac{L}{w^2} \tag{47}$$

hence

$$\frac{dL}{d\phi} = \frac{3\mu}{2Lw} R^2 J_2 \sin 2\phi \tag{48}$$

Note that in this relation L is not a constant.

Using Eq. (47) we can derive an equation for w as a function of ϕ:

$$\frac{w''}{w} - 2\left(\frac{w'}{w}\right)^2 + \frac{3\mu R^2 J_2 \sin 2\phi}{2L^2 w^2} w' = 1 - \frac{\mu}{2L^2 w}(2w^2 - 9R^2 J_2 \cos^2\phi + 3R^2 J_2) \tag{49}$$

where the primes denote differentiation with respect to ϕ.

Imposing a change of variable to $u = 1/w$, then Eq. (49) transforms into

$$u'' + u - \frac{\mu}{L^2} = \frac{3\mu R^2 J_2}{2L^2}[(1 - 3\cos^2\phi)u^2 - uu' \sin 2\phi] \tag{50}$$

This equation corrects the erroneous subsection on polar orbits in Humi [20].

For realistic spheroids in the solar system, such as the Earth, $J_2 \ll 1$. We therefore set $\epsilon = J_2$ as a small parameter and approximate the solution to Eqs. (46) and (50) by a first-order perturbation expansion in this parameter. In this setting

$$u = u_0 + \epsilon u_1, \qquad L^2 = L_0 + \epsilon L_1$$

and we obtain to order ϵ

$$L^2 = h^2 + 3\epsilon\mu R^2 \int u_0 \sin 2\phi \, d\phi = h^2 + 3\epsilon\mu R^2 I \tag{51}$$

where I represents the indefinite integral in Eq. (51).

For u_0 and u_1, we have

$$u_0'' + u_0 - \frac{\mu}{h^2} = 0 \tag{52}$$

$$u_1'' + u_1 + \frac{3\mu^2 R^2}{h^4} I = \frac{3\mu R^2}{h^2}\left[(1 - 3\cos^2\phi)u_0^2 + u_0 u_0' \sin 2\phi\right] \tag{53}$$

Assume that at time zero $\phi = \phi_0$ and the initial conditions on u, u' are $u(\phi_0) = U_0$, $u'(\phi_0) = U_0'$. The corresponding initial conditions on u_0, u_1 will then be

$$u_0(\phi_0) = U_0, \qquad u_0'(\phi_0) = U_0', \qquad u_1(\phi_0) = u_1'(\phi_0) = 0$$

hence

$$u_0 = \frac{\mu}{h^2}(1 + e\cos(\phi - \psi)) \tag{54}$$

where e and ψ are constants that are determined from the initial conditions on u_0 and u_0'. Substituting this result into Eq. (51) leads to

$$I = -\frac{\mu}{2h^2}\left\{\cos 2\phi + e\left[\cos(\phi - \psi) + \frac{1}{3}\cos(3\phi - \psi)\right]\right\} \tag{55}$$

The general solution for u_1 is

$$u_1 = C_1 \cos\phi + C_2 \sin\phi + u_p \tag{56}$$

where u_p is a particular solution of Eq. (53). It can be found by standard methods:

$$u_p = \frac{\mu^3 R^2}{h^6}\left(\cos 2\phi - \frac{3}{2} + e u_{p1} + e^2 u_{p2}\right) \tag{57}$$

where

$$u_{p1} = \tfrac{1}{16}[5\cos(3\phi - \phi_0) - 12\cos(\phi - \phi_0) - 24\phi\sin(\phi - \phi_0) - 36\phi\sin(\phi + \phi_0) - 18\cos(\phi + \phi_0)],$$

$$u_{p2} = \tfrac{1}{40}[10\cos 2(\phi - \phi_0) + \cos(4\phi - 2\phi_0) - 75\cos(2\phi_0) + 30\cos(2\phi) - 30]$$

Although this solution is cumbersome, it simplifies drastically for $e = 0$:

$$u_p = \frac{\mu^3 R^2}{h^6}\left(\cos 2\phi - \frac{3}{2}\right) \tag{58}$$

Applying the initial conditions to u_1 for $e = 0$ we then have

$$u_1 = \frac{\mu^3 R^2}{2h^6}[2\cos 2\phi + (1 + 4\cos^2\phi_0)\cos(\phi - \phi_0) - 4\cos(\phi + \phi_0) - 3] \tag{59}$$

III. Linearized Relative-Motion Equations

If the positions of the satellite and spacecraft are denoted by $\boldsymbol{\rho}$, $\boldsymbol{\rho} c$ then their respective equations of motion are

$$\ddot{\boldsymbol{\rho}} = -\nabla U(\boldsymbol{\rho}), \qquad \ddot{\boldsymbol{\rho}}_c = -\nabla U(\boldsymbol{\rho}_c) + \mathbf{T}/m \tag{60}$$

where $U(\boldsymbol{\rho})$ is given by Eq. (1), $\mathbf{T}$ is the applied force from the spacecraft thrusters, m is the mass of the spacecraft, and the dots represent differentiation with respect to time. The relative position of the spacecraft with respect to the satellite is $\mathbf{s} = \boldsymbol{\rho}_c - \boldsymbol{\rho}$. This leads to the equation

$$\ddot{\boldsymbol{\rho}} + \ddot{\mathbf{s}} = -\nabla U(\boldsymbol{\rho}_c) + \mathbf{T}/m \tag{61}$$

Using Eq. (60) it can be rewritten as

$$\ddot{\mathbf{s}} = \nabla U(\boldsymbol{\rho}) - \nabla U(\boldsymbol{\rho}_c) + \mathbf{T}/m \tag{62}$$

Assuming that $|\mathbf{s}| \ll \rho$, we can approximate $\nabla U(\boldsymbol{\rho}_c) = \nabla U(\boldsymbol{\rho} + \mathbf{s})$ by a first-order Taylor polynomial in $\mathbf{s}$. After some algebra, this leads to the following linear equation of motion of the spacecraft with respect to the satellite in the inertial coordinate system attached to the spheroid's center, specifically

$$\ddot{\mathbf{s}} = \mathbf{F}_c \tag{63}$$

where

$$\mathbf{F}_c = -\frac{\mu}{\rho^3}\mathbf{s} + \frac{3\mu}{\rho^5}(\boldsymbol{\rho}\cdot\mathbf{s})\boldsymbol{\rho} + \mu R^2 J_2\left[\frac{15z}{\rho^7}\boldsymbol{\rho}\cdot\mathbf{s} - \frac{3s_3}{\rho^5}\right]\mathbf{k} + \mu R^2 J_2\left\{\left[\frac{21}{2\rho^9}\boldsymbol{\rho}\cdot\mathbf{s}(\rho^2 - 5z^2) - \frac{3}{\rho^7}(\boldsymbol{\rho}\cdot\mathbf{s} - 5zs_3)\right]\boldsymbol{\rho} - \frac{3}{2\rho^7}(\rho^2 - 5z^2)\mathbf{s}\right\} + \mathbf{T}/m \tag{64}$$

In a coordinate system rotating with the satellite, the relative-motion equation (63) becomes [21]

$$\ddot{s} + 2\boldsymbol{\Omega}\times\dot{\mathbf{s}} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{s}) + \dot{\boldsymbol{\Omega}}\times\mathbf{s} = F_c \tag{65}$$

where $\boldsymbol{\Omega}$ is the orbital angular velocity of the satellite.

IV. Reduction of the Relative-Motion Equations

To reduce Eq. (65) for equatorial and near-equatorial satellite orbits, we now introduce an orthogonal coordinate system that is attached to the satellite so that the unit vector $\mathbf{e}_y$ is in the direction of $\boldsymbol{\rho}$. The unit vector $\mathbf{e}_x$ (along the positive x direction) is orthogonal to $\mathbf{e}_y$ in the instantaneous orbital plane with a direction opposite to the motion of the satellite. Finally, the unit vector $\mathbf{e}_z$ completes a right-handed system. See Fig. 1b.

In this system $\mathbf{s} = (x_1, x_2, x_3)$ and $\boldsymbol{\rho} = (0, \rho, 0)$. Moreover, for a satellite in equatorial or near-equatorial orbit, according to the approximations made in the previous section, $\boldsymbol{\Omega} = (0, 0, \dot{\theta}) = (0, 0, \omega)$. For brevity we drop the force due to the thrusters in $\mathbf{F}_c$.

A. Equatorial Orbits

In an equatorial orbit, the motion of the satellite is in the plane $z = 0$ and the expression for $\mathbf{F}_c$ without thrusting reduces to

$$\mathbf{F}_c = \left(-\frac{\mu s_1}{\rho^3} - \frac{C s_1}{\rho^5}, \frac{2\mu s_2}{\rho^3} + \frac{4 C s_2}{\rho^5}, -\frac{\mu s_3}{\rho^3} - \frac{3 C s_3}{\rho^5}\right) \tag{66}$$

where $C = \frac{3\mu R^2 J_2}{2}$. Using Eq. (26) and the fact that $\rho = (\frac{L}{\omega})^{1/2}$, the relative-motion equation (65) becomes

$$\begin{aligned}\ddot{x}_1 &= (\omega^2 - k_1\omega^{3/2} - k_2\omega^{5/2})x_1 + \dot{\omega}x_2 + 2\omega\dot{x}_2 \\ \ddot{x}_2 &= (\omega^2 + 2k_1\omega^{3/2} + 4k_2\omega^{5/2})x_2 - \dot{\omega}x_1 - 2\omega\dot{x}_1 \\ \ddot{x}_3 &= -(k_1\omega^{3/2} + 3k_2\omega^{5/2})x_3\end{aligned} \tag{67}$$

Changing the independent variable from t to θ in these equations we obtain

$$\begin{aligned}\omega^2 x_1'' + \omega\omega' x_1' &= (\omega^2 - k_1\omega^{3/2} - k_2\omega^{5/2})x_1 + \omega\omega' x_2 + 2\omega^2 x_2' \\ \omega^2 x_2'' + \omega\omega' x_2' &= (\omega^2 + 2k_1\omega^{3/2} + 4k_2\omega^{5/2})x_2 - \omega\omega' x_1 - 2\omega^2 x_1' \\ \omega^2 x_3'' + \omega\omega' x_3' &= -(k_1\omega^{3/2} + 3k_2\omega^{5/2})x_3\end{aligned} \tag{68}$$

We now introduce new variables

$$y_i = \omega^{1/2} x_i, \qquad i = 1, 2, 3 \tag{69}$$

This leads to the following system of equations:

$$\begin{aligned}y_1'' - M(\omega)y_1 &= 2y_2' \\ y_2'' - M(\omega)y_2 &= -2y_1' - [3k_1\omega^{-1/2} + 5k_2\omega^{1/2}]y_2 \\ y_3'' - I(\omega)y_3 &= -2k_2\omega^{1/2}y_3\end{aligned} \tag{70}$$

where

$$M(\omega) = I(\omega) + 1 \tag{71}$$

$$I(\omega) = \frac{-1}{4}\omega^{-2}(\omega')^2 - \frac{1}{2}\omega^{-1}\omega'' + k_1\omega^{-1/2} + k_2\omega^{1/2} \tag{72}$$

We now present a remarkable simplification. Making the substitution $\omega = Lu(\theta)^2$ in Eq. (72), we obtain

$$I(\omega) = \frac{-1}{u}[-u'' + k_1 L^{-1/2} + k_2 L^{1/2} u^2] \tag{73}$$

From Eq. (27) it then follows that $I(\omega) = -1$. The relative-motion equations (70) reduce, therefore, to the concise set

$$y_1'' = 2y_2' + \frac{T_1}{m\omega^{3/2}} \tag{74}$$

$$y_2'' = -2y_1' - [3k_1\omega^{-1/2} + 5k_2\omega^{1/2}]y_2 + \frac{T_2}{m\omega^{3/2}} \tag{75}$$

$$y_3'' = -(1 + 2k_2\omega^{1/2})y_3 + \frac{T_3}{m\omega^{3/2}} \tag{76}$$

The reader will observe that we have restored the expressions for the forcing functions due to the thrusters in these equations.

We note that Eq. (76) is independent of Eqs. (74) and (75). Moreover, when $\mathbf{T} = 0$ identically, Eqs. (74) and (75) can be reduced to one equation by integrating Eq. (74). This leads to the equation

$$y_2'' + [4 + 3k_1\omega^{-1/2} + 5k_2\omega^{1/2}]y_2 = \alpha \tag{77}$$

where α is an integration constant. In these equations, the effect of the J_2 term is represented explicitly by the terms containing k_2 and implicitly by its impact on ω, or equivalently the orbit of the satellite.

As noted above, ω can be expressed, in general, in terms of elliptic functions. In the special case where the satellite is in circular orbit, then Eqs. (74–76) reduce to a system with constant coefficients that has a closed-form solution in terms of elementary functions. We shall present this solution subsequently.

Equations (74–77) are natural generalizations of the rendezvous equations for a spacecraft with a satellite in Keplerian orbit around a spherical Earth (or a spherical body in general). The form and conciseness are reminiscent of the Tschauner–Hempel equations [4,7]. If the orbit is circular then ω is constant, and these equations take a form similar to the Clohessy–Wiltshire equations [2].

B. Near-Equatorial Orbits

We now discuss the case in which the angle of inclination of the satellite plane with respect to the equatorial plane is small for the entire arc of the orbit under consideration. For this case we approximate the expression for $\mathbf{F}_c$ by neglecting terms in $(\frac{z}{\rho})^2$ and identify the direction of e_z with $\mathbf{k}$. In the coordinate system attached to the satellite that was introduced at the beginning of this section, the expression for the force in the absence of thrust takes the form

$$\mathbf{F}_c = \left(-\frac{\mu s_1}{\rho^3} - \frac{C s_1}{\rho^5}, \frac{2\mu s_2}{\rho^3} + \frac{4 C s_2}{\rho^5} + \frac{10 C z s_3}{\rho^6}, -\frac{\mu s_3}{\rho^3} - \frac{3 C s_3}{\rho^5} + \frac{10 C z s_2}{\rho^6}\right) \tag{78}$$

Following exactly the same steps as in the previous section and including the thrust term, the relative-motion equations take the following form:

$$y_1'' = 2y_2' + \frac{T_1}{m\omega^{3/2}} \tag{79}$$

$$y_2'' = -2y_1' - [3k_1\omega^{-1/2} + 5k_2\omega^{1/2}]y_2 + 10k_2 L^{-1/2}\omega z y_3 + \frac{T_2}{m\omega^{3/2}} \tag{80}$$

$$y_3'' = -(1 + 2k_2\omega^{1/2})y_3 + 10k_2 L^{-1/2}\omega z y_2 + \frac{T_3}{m\omega^{3/2}} \tag{81}$$

We see that in the case when $\mathbf{T} = \mathbf{0}$ identically, we obtain a coupled system of three equations that can be reduced to two. For this problem we have explicit dependence of both z (i.e., the z coordinate of the satellite position) and ω on θ. If the approximations leading to Eq. (41) hold, then we can use Eqs. (30) and (41) to express z explicitly in terms of θ in these equations.

C. Solution of the Relative-Motion Equations for Equatorial Circular Orbit

We have shown that there are circular orbits in the equatorial plane having a radius and angular velocity given respectively by Eqs. (35) and (36). For intervals where the spacecraft is not thrusting (i.e., $\mathbf{T} = \mathbf{0}$ identically), the Eqs. (74–76) describe relative motion about a satellite in circular orbit. Because ω is constant in this case, the Eqs. (74–76) simplify and can be solved in closed form. These equations are very similar to the well-known Clohessy–Wiltshire equations [2].

Using Eq. (26) to define the constants k_1 and k_2 and Eq. (36) for ω_0, Eq. (77) becomes

$$y_2'' + k^2 y_2 = \alpha \tag{82}$$

where

$$k^2 = 4 + \frac{3\mu}{L^2}\left(r_0 + \frac{5\mu R^2 J_2}{2r_0}\right) \tag{83}$$

and Eq. (76) is written as

$$y_3'' + w^2 y_3 = 0 \tag{84}$$

where

$$w^2 = 1 + \frac{\mu R^2 J_2}{r_0 L^2} \tag{85}$$

The integration of Eqs. (82) and (84) is straightforward. With T_1 identically zero, the solution of Eq. (74) is also straightforward from the solution of y_2. The complete solution of the homogeneous form of Eqs. (74–76) is therefore

$$y_1 = \frac{2C_1}{k}\sin(k\theta + \phi) + \left(\frac{2}{k^2} - \frac{1}{2}\right)\alpha\theta + C_2 \tag{86}$$

$$y_2 = C_1\cos(k\theta + \phi) + \frac{\alpha}{k^2} \tag{87}$$

$$y_3 = C_3\cos(w\theta + \psi) \tag{88}$$

where C_1, C_2, C_3, α, ϕ, and ψ are arbitrary constants. One observes that Eq. (88) describes simple harmonic motion outside the equatorial plane. Depending on the initial conditions, the curve defined by Eqs. (86–88) can take a variety of shapes, helical, cycloidal, elliptical, etc., which are similar to the shapes from the solution of the Clohessy–Wiltshire equations, except that the constants k and ω produce a higher frequency oscillation from the J_2 effects. These effects generalize the solution curves that can take very different shapes from the Clohessy–Wiltshire solution curves if J_2 is sufficiently large. When $J_2 = 0$ these solution curves reduce to the Clohessy–Wiltshire solution curves, whose shapes are discussed in Carter [23].

D. Polar Orbits

For satellites in polar orbits, we introduce an orthogonal coordinate system that is attached to the satellite so that the unit vector $\mathbf{e}_x$ is in the direction of $\boldsymbol{\rho}$. The unit vector $\mathbf{e}_z$ (along the positive z direction) is orthogonal to $\mathbf{e}_x$ in the orbital plane with a direction opposite to the motion of the satellite. Finally, the unit vector $\mathbf{e}_y$ completes a right-handed system. In this coordinate system $\boldsymbol{\rho} = (w, 0, 0)$ and $\mathbf{s} = (x_1, x_2, x_3)$. The components of the force $\mathbf{F}$ along $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, respectively, are

$$F_x = \left(\frac{2\mu}{w^3} + \frac{6\mu R^2 J_2}{w^5}\right)s_1 \tag{89}$$

$$F_y = \left(-\frac{\mu}{w^3} + \frac{3\mu R^2 J_2}{2w^5}\right)s_2 \tag{90}$$

$$F_z = \left(-\frac{\mu}{w^3} - \frac{9\mu R^2 J_2}{2w^5}\right)s_3 \tag{91}$$

Following the same steps that were made to reduce the equatorial relative-motion equations and using Eqs. (48) and (50), we obtain after many algebraic manipulations the following relative-motion equations for $y_i = \omega(\phi)^{1/2}x_i$, $i = 1, 2, 3$:

$$y_1'' + 2y_3' - \frac{3\mu}{L^2 u}y_1 = \frac{\mu R^2 J_2}{L^2}\left[\left(-6 + \frac{3}{2}\cos^2\phi\right)u - \frac{3u'}{4}\sin 2\phi + \frac{27\mu R^2 J_2}{16L^2}\sin^2 2\phi\right]y_1 \tag{92}$$

$$y_2'' + y_1 = \frac{\mu R^2 J_2}{L^2}\left[-\frac{3u}{2}\left(\frac{3}{2} + \cos^2\phi\right) - \frac{3u'}{4}\sin 2\phi + \frac{27\mu R^2 J_2}{16L^2}\sin^2 2\phi\right]y_2 \tag{93}$$

$$y_2'' - 2y_1' = \frac{\mu R^2 J_2}{L^2}\left[-\frac{3u}{2}\sin^2\phi - \frac{3u'}{4}\sin 2\phi + \frac{27\mu R^2 J_2}{16L^2}\sin^2 2\phi\right]y_3 \tag{94}$$

Observe that in these equations L is not a constant. Clearly these equations lack the conciseness of the equatorial or near-equatorial relative-motion equations.

V. Model Validation

As a first step toward testing the accuracy of our model, we evaluated the impact of the J_2 term on the orbit of satellites in the equatorial plane with orbits of different heights above the Earth. We considered orbits with $\rho = 6700$, 6900, 7100, 7300 km and used initial conditions that correspond to circular Keplerian orbits (around a spherical body). We assumed the value of J_2 for the Earth to be 0.00108263. We found that the addition of the J_2 term changed ρ by less than 1 km.

To test the accuracy of replacing Eqs. (6) and (7) with Eqs. (25) and (40) for near-equatorial satellites, we simulated these two sets of equations with the same initial conditions: $r = 7100$ km, $v = 0.01$, and $\dot{\theta} = 0.0011$ rad/ sec. The resulting trajectories for r and z, respectively, using Eqs. (6) and (7) are plotted in Figs. 2 and 3. The deviations from this orbit that result from using Eqs. (25) and (40) over 20 revolutions are plotted in Figs. 4 and 5. Under these conditions, the deviations in r, z do not exceed 1 and 0.01 km, respectively.

In Fig. 6, we display the ratio of the angular momentum components L_θ/L_z for the satellite whose orbit was computed in Figs. 2 and 3. This plot justifies the approximation $\mathbf{L} = r^2\dot{\theta}\mathbf{k}$ for satellites in near-equatorial orbit.

To gauge the relative importance of the terms containing k_2 that are due to the J_2 term in the gravitational field on the rendezvous problem, we simulated Eqs. (74–76) with and without these terms using the same initial conditions and equatorial satellites that are in circular orbits. The deviation in y_2 due to this term is presented in Fig. 7. The same comparison was carried out for satellites in near-equatorial orbits. The plot for the deviations in y_2 is similar to Fig. 7. The deviations in y_3 are presented in Fig. 8.

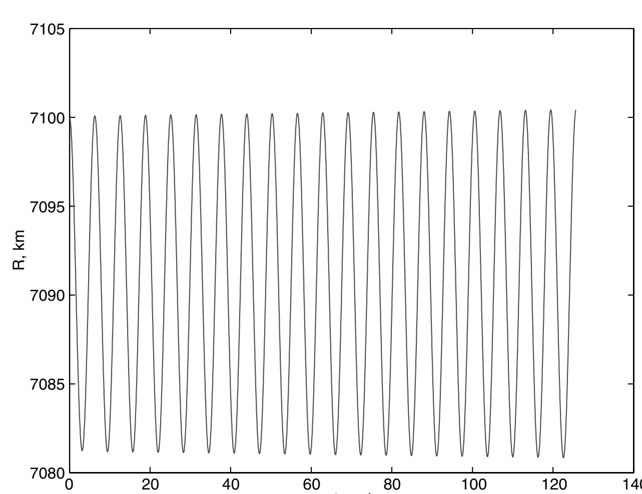

Fig. 2 The trajectory for r as a function of θ using Eqs. (6) and (7) with the initial conditions given in the text.

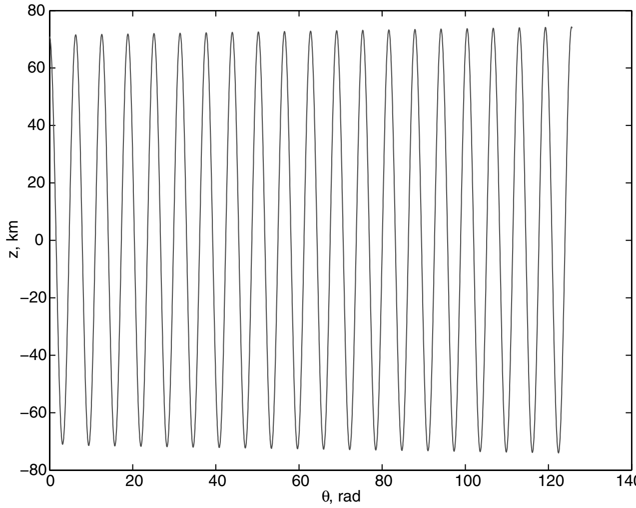

Fig. 3 Same as Fig. 2 for *z*.

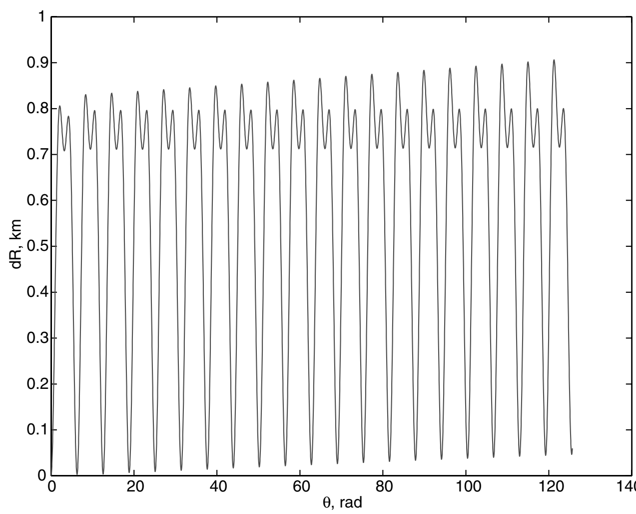

Fig. 4 The resulting deviations in *r* when Eqs. (25) and (40) are used to compute the trajectory.

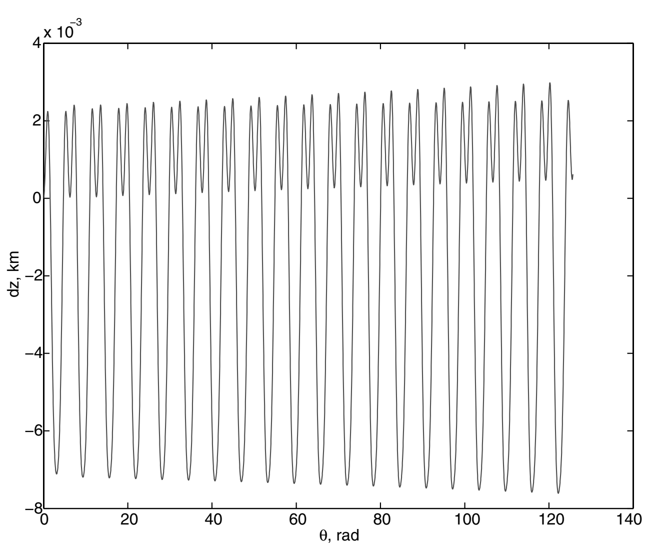

Fig. 5 Same as Fig. 5 for the deviations in *z*.

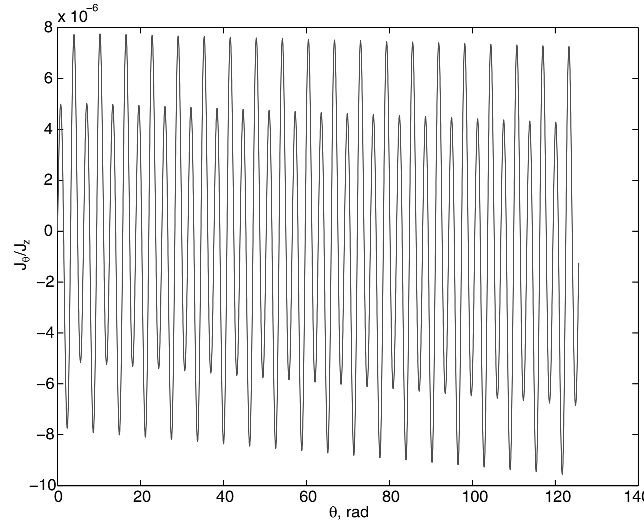

Fig. 6 The ratio $\frac{L_\theta}{L_z}$ for the orbit in Figs. 2 and 3.

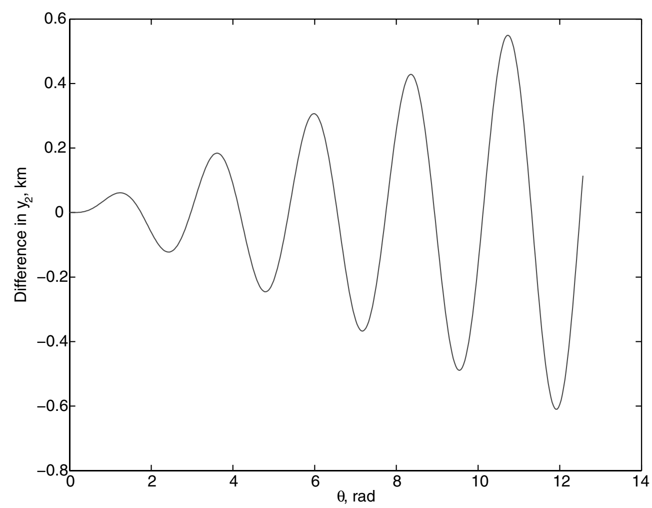

Fig. 7 The impact of the J_2 term in Eq. (77) on the solution of y_2 when the satellite is in equatorial orbit.

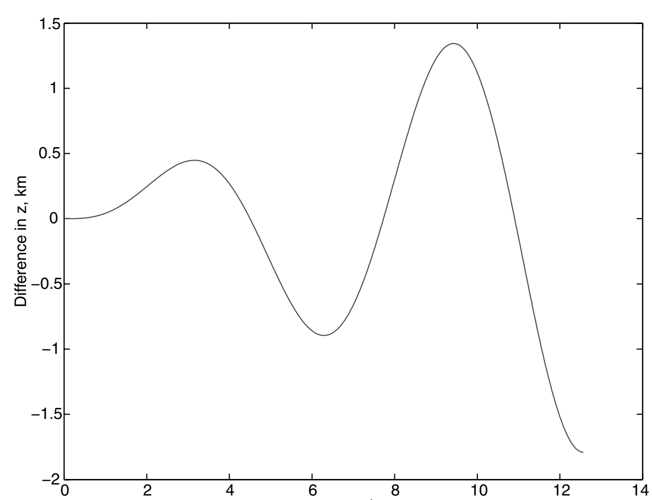

Fig. 8 The impact of the J_2 term in Eqs. (80) and (81) on the solution of y_3 when the satellite is in near-equatorial orbit.

VI. Fuel-Optimal Rendezvous of a Spacecraft

We consider here briefly several fuel-optimal rendezvous problems for a spacecraft with a satellite in equatorial orbit.

A. Constant-Mass Bounded-Thrust Problem

For this problem, we assume that the magnitude of the thrust is bounded and the mass loss to the spacecraft during the rendezvous maneuvers is small and can be considered negligible. The treatment here is similar to the ones in Carter and Humi [7] and Humi [8]. Assuming that the rate of fuel consumption is proportional to the magnitude of the thrust $\mathbf{T}(t)$, the fuel-optimal rendezvous problem can be formulated as one that minimizes

$$I(\mathbf{T}) = \int_{t_0}^{t_f} |\mathbf{T}(t)|\,\mathrm{d}t = \int_{\theta_0}^{\theta_f} \frac{|\mathbf{T}(\theta)|}{\omega}\,\mathrm{d}\theta \tag{95}$$

with initial and terminal conditions

$$\mathbf{y}(\theta_0) = \mathbf{y}_0, \qquad \mathbf{y}'(\theta_0) = \mathbf{v}_0 \tag{96}$$

$$\mathbf{y}(\theta_f) = \mathbf{y}_f, \qquad \mathbf{y}'(\theta_f) = \mathbf{v}_f \tag{97}$$

and subject to the differential equations

$$\mathbf{y}' = \mathbf{v} \qquad \mathbf{v}' = A(\theta)\mathbf{y} + B(\theta)\mathbf{v} + b\omega^{-3/2}\mathbf{u} \tag{98}$$

In this formulation, $\mathbf{u}(\theta) = \frac{\mathbf{T}(\theta)}{T_{\max}}$ is the normalized thrust where $T_{\max}$ is the maximum thrust level so that $|\mathbf{u}| \le 1$, and

$$A(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & G_1 & 0 \\ 0 & 0 & G_2 \end{pmatrix}, \qquad B(\theta) = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{99}$$

where

$$G_1 = -(3k_1\omega^{-1/2} + 5k_2\omega^{1/2}), \qquad G_2 = -(1 + 2k_2\omega^{1/2}) \tag{100}$$

and the mass m is absorbed in the constant b.

Because the equations of motion for this system are linear, the Pontrayagin principle provides necessary and sufficient conditions for optimality. The Hamiltonian of the system is

$$H(\mathbf{p},\mathbf{q},\mathbf{w},\lambda) = \frac{\lambda|\mathbf{w}|}{\omega} + \mathbf{p}(\theta)^T\cdot\mathbf{v}(\theta) + \mathbf{q}(\theta)^T\cdot[A(\theta)\mathbf{y} + B(\theta)\mathbf{v} + b\omega^{-3/2}\mathbf{w}] \tag{101}$$

where $\lambda \ge 0$ and $\mathbf{p}$, $\mathbf{q}$ satisfy

$$\mathbf{p}' = \frac{\partial H}{\partial \mathbf{y}} = -A^T(\theta)\mathbf{q}(\theta) \tag{102}$$

$$\mathbf{q}' = \frac{\partial H}{\partial \mathbf{v}} = -\mathbf{p} - B^T(\theta)\mathbf{q}(\theta) \tag{103}$$

and cannot be identically zero.

For normal solutions, λ is nonzero and is arbitrary; we set $\lambda = b$. The optimal control $\mathbf{u}(\theta)$ must then minimize the expression

$$\frac{b}{\omega}\left[|\mathbf{w}| + \frac{(\mathbf{q}^T\cdot\mathbf{w})}{\omega^{1/2}}\right] \tag{104}$$

Hence, it follows that

$$\mathbf{u}(\theta) = \begin{cases} 0, & \frac{|\mathbf{q}(\theta)|}{\omega^{1/2}} < 1 \\ -\frac{\mathbf{q}(\theta)}{|\mathbf{q}(\theta)|}, & \frac{|\mathbf{q}(\theta)|}{\omega^{1/2}} > 1 \end{cases} \tag{105}$$

and the primer vector $\mathbf{q}$ satisfies the differential equation

$$\mathbf{q}'' - B\mathbf{q}' - A\mathbf{q} = 0 \tag{106}$$

which is equivalent to Eqs. (98) with $\mathbf{u} = 0$. If $\frac{|\mathbf{q}(\theta)|}{\omega^{1/2}} = 1$ at infinitely many points on a bounded interval, then it is identically one and the optimal solution is *singular*. This can occur only if $\mathbf{q}(\theta)\cdot\mathbf{q}(\theta) = \omega(\theta)$ identically.

The treatment of the fuel-optimal rendezvous problem of a spacecraft with a satellite in near-equatorial orbit is similar to the aforementioned treatment.

B. Varying-Mass Bounded-Thrust Problem

When the mass loss of the spacecraft during the rendezvous maneuvers has to be taken into account, the objective is again to minimize Eq. (95) subject to Eqs. (96–98), but we also have in addition that

$$m'(\theta) = -\frac{k_0|\mathbf{u}(\theta)|}{\omega} \tag{107}$$

where $m(\theta)$ represents the mass of the spacecraft, k_0 is a constant that depends on the specific impulse of the propellant, and again $\mathbf{u}(\theta) = \frac{\mathbf{T}(\theta)}{T_{\max}}$ with $|u| \le 1$ as in the previous problem. The treatment is similar to that in Carter [24]. In this case the resulting control problem is nonlinear and the Pontrayagin principle provides only the necessary conditions for solutions of this problem. The expression for the Hamiltonian is

$$H(\mathbf{p},\mathbf{q},\mathbf{w},\lambda) = \frac{\lambda|\mathbf{w}|}{\omega} + \mathbf{p}(\theta)^T\cdot\mathbf{v}(\theta) + \mathbf{q}^T\cdot\left[A(\theta)\mathbf{y} + B(\theta)\mathbf{v} + \frac{b\omega^{-3/2}(\theta)\mathbf{w}}{m(\theta)}\right] - \frac{k_0 r(\theta)|\mathbf{w}|}{\omega} \tag{108}$$

where the mass $m(\theta)$ is not absorbed in the constant b, contrary to the previous problem. Again, $\lambda \ge 0$ and the conjugate variables $\mathbf{p}$, $\mathbf{q}$ must satisfy Eqs. (102) and (103), whereas $r(\theta)$ must satisfy

$$r'(\theta) = \frac{b\omega^{-3/2}}{m(\theta)^2}\mathbf{q}(\theta)^T\cdot\mathbf{u}(\theta) \tag{109}$$

Again, the primer vector $\mathbf{q}(\theta)$ must be defined by Eq. (106) and cannot be identically zero. Viewed as a function of $\mathbf{w}$, the Hamiltonian is at minimum when $\mathbf{u}$ minimizes the expression

$$[\lambda - k_0 r(\theta)]|\mathbf{w}| + \frac{b\omega^{-1/2}}{m(\theta)}\mathbf{q}(\theta)^T\cdot\mathbf{w} \tag{110}$$

This happens when

$$\mathbf{u} = -\frac{q(\theta)}{|q(\theta)|}f(\theta) \tag{111}$$

where

$$f(\theta) = \begin{cases} 0, & s(\theta) > 0 \\ 1, & s(\theta) < 0 \end{cases} \tag{112}$$

and

$$s(\theta) = \lambda - k_0 r(\theta) - \frac{b\omega^{-1/2}}{m(\theta)}|\mathbf{q}(\theta)| \tag{113}$$

is called the switching function. The switching function cannot equal zero at infinitely many points on a bounded interval without being identically zero. If the switching function is identically zero, then the minimizing solution is singular.

C. Power-Limited Rendezvous

For power-limited thrusters (i.e., electric or nuclear), the rate of fuel consumption is assumed to be proportional to the square of the magnitude of the thrust. For this class of problems, we assume that the mass expended is negligible compared with the mass of the spacecraft, and we seek to minimize

$$I(T) = \int_{t_0}^{t_f} \frac{1}{2} |\mathbf{T}(t)|^2 \, \mathrm{d}t = \int_{\theta_0}^{\theta_f} \frac{1}{2} \frac{|\mathbf{T}(\theta)|^2}{\omega} \, \mathrm{d}\theta \tag{114}$$

subject to the boundary conditions (96) and (97) and the differential equations (98) where the matrices $A(\theta)$ and $B(\theta)$ are defined through Eqs. (99) and (100) and, in this case, $u(\theta) = T(\theta)$. The treatment of this problem is similar to that of Carter [25].

For this case also the Pontrayagin principle provides both the necessary and sufficient conditions for optimality. The Hamiltonian is

$$H(\mathbf{p}, \mathbf{q}, \mathbf{w}, \lambda) = \frac{\lambda |\mathbf{w}|^2}{2\omega} + \mathbf{p}(\theta)^T \cdot \mathbf{v}(\theta) + \mathbf{q}(\theta)^T \cdot [A(\theta)\mathbf{y} + B(\theta)\mathbf{v} + b\omega^{-3/2}\mathbf{w}] \tag{115}$$

where $\lambda \geq 0$ and $\mathbf{p}$, $\mathbf{q}$ again satisfy Eqs. (102) and (103), leading to the differential equation (106) that describes the primer vector $\mathbf{q}$. Again, the mass m is absorbed in the constant b. For normal solutions, we again set $\lambda = b$. It is seen that the minimizing control $\mathbf{u}(\theta)$ must minimize the expression

$$\frac{1}{2} \frac{|\mathbf{w}|^2}{\omega} + \frac{(\mathbf{q}^T \cdot \mathbf{w})}{\omega^{3/2}} \tag{116}$$

Therefore, the normal optimal control function is given by

$$\mathbf{u}(\theta) = -\omega^{-1/2} \mathbf{q}(\theta) \tag{117}$$

D. Impulsive Rendezvous

If the bounded-thrust problem of Sec. VI.A has a very high bound $T_{\max}$, it may be preferable to model the problem as an impulsive problem [26,27]. This is an idealization in which the momentum change of a spacecraft resulting from a short-duration burn of the engine thrusters is treated as an instantaneous jump without a corresponding discontinuity of the position of the spacecraft. To follow the methodology of Carter and Brient [26], one first finds fundamental matrix solutions associated with the homogeneous form of Eqs. (98) and also of the system (102) and (104) and applies Theorem 3.2 of that reference. Because it is unlikely that such fundamental matrix solutions can be found in closed form, we adapt that Theorem and state it in terms of the differential equations (98) and the primer vector $\mathbf{q}$ defined through Eq. (106) instead. This adaptation of Theorem 3.2 presents the results in a form very similar to that of Prussing [27] and of earlier papers mentioned in the references of these two papers.

We state the impulsive minimization problem as follows.

Given a positive integer k where $2 \leq k \leq 6$ and a closed interval $\theta_0 \leq \theta \leq \theta_f$, we seek a finite subset $K = \{\theta_1, \ldots, \theta_k\}$ of this interval and velocity increments $\{\Delta\mathbf{v}_1, \ldots, \Delta\mathbf{v}_k\} \varepsilon R^3$ that minimize the total characteristic velocity

$$I(\Delta\mathbf{v}_1, \ldots, \Delta\mathbf{v}_k, \theta_1, \ldots, \theta_k) = \sum_{i=1}^{k} \frac{|\Delta\mathbf{v}_i|}{\omega(\theta_i)} \tag{118}$$

subject to the boundary conditions (96) and (97) and the differential equations (98) without the term containing the forcing function $\mathbf{u}$. The equations are valid everywhere on the interval $\theta_0 \leq \theta \leq \theta_f$ except on the finite subset K, where the velocity increments can be applied. Again, the primer vector $\mathbf{q}$ is defined as a solution of the differential equation

$$\mathbf{q}'' - B\mathbf{q}' - A\mathbf{q} = 0 \tag{119}$$

Adapting previous results [26,27] we have the following.

The necessary and sufficient conditions for the finite subset $K = \{\theta_1, \ldots, \theta_k\}$ and the velocity increments $\{\Delta\mathbf{v}_1, \ldots, \Delta\mathbf{v}_k\}$ to be a minimizing solution of the impulsive minimization problem are that the boundary conditions (96) and (97) are satisfied, the differential equations (98) without the forcing term containing $\mathbf{u}$ are satisfied except on the subset K, the primer vector $\mathbf{q}$ satisfies Eq. (119), and

$$\Delta\mathbf{v}_i = 0, \quad \text{or} \quad \frac{\Delta\mathbf{v}_i}{|\Delta\mathbf{v}_i|} = -\mathbf{q}(\theta_i) = 1, \qquad i = 1, \ldots, k \tag{120}$$

$$\theta_i = \theta_0, \quad \text{or} \quad |\mathbf{q}(\theta_i)|' = 0, \quad \text{or} \quad \theta_i = \theta_f, \qquad i = 1, \ldots, k \tag{121}$$

$$|\mathbf{q}(\theta)| \leq 1, \qquad \theta_0 \leq \theta \leq \theta_f \tag{122}$$

VII. Conclusions

Satellite orbits in a noncentral gravitational field about an oblate body which contains the J_2 term are very different from those about a point mass or a homogeneous sphere. Considerations drawn from the angular momenta showed that the orbits are, in general, not planar, the only exceptions being equatorial and polar orbits. With the exception of linear motion along the polar axes or radially in the intersection of the polar and equatorial planes, the magnitude of the angular momentum of a polar orbit is never constant. On the other hand, every orbit in the equatorial plane has constant angular momentum. There are no Keplerian orbits except circular orbits, and these exist only in the equatorial plane.

Equations of motion of a satellite were examined for equatorial orbits, near-equatorial orbits, and polar orbits. These equations were most concise for the equatorial orbits, resulting in an implicit integral and a solution for the true anomaly in terms of elliptic functions. For a given angular momentum, we found that there can be a unique equatorial circular orbit if and only if $J_2 \leq 2/3$, a condition not violated by any planet or moon in the solar system. Approximate equations of motion for near-equatorial orbits have solutions that can be expressed in terms of special functions. Approximate solutions for the equations of motion for polar orbits can be determined through a first-order perturbation expansion, but their algebraic expressions are somewhat cumbersome.

We also considered the problem of the relative motion of a spacecraft with a satellite in orbit in the noncentral gravitational field of a spheroid that contains the J_2 term. We derived a linearized system of equations in terms of the relative motion of the spacecraft with respect to the satellite for equatorial, near-equatorial, and polar orbits.

If the orbit of the satellite is equatorial, we found appropriate transformations that effectively reduced these linearized equations to two second-order differential equations. It was found that the only data needed to solve these equations other than the initial conditions is the angular velocity of the satellite. These equations exhibit a remarkable similarity in form and conciseness to the Tschauner–Hempel equations. For the special case of a circular orbit, the relative-motion equations simplify enough to be solvable in closed form. We found analytic solutions having the same general geometric shapes as the Clohessy–Wiltshire solutions, but there are some differences due to the J_2 effect.

We also derived the relative-motion equations of a spacecraft with a satellite in near-equatorial orbit. It was found that, as in the previous case, these equations can be reduced to two second-order equations, although, in this case, the equations are coupled.

The relative-motion equations for a satellite in polar orbit were more cumbersome than the previous cases.

We tested some of the equations and the various approximations with several simulations whose results were presented in the figures. The simulations confirm the validity of the approach presented in this study.

The relative-motion equations presented herein can be used as a basis for fuel-optimal rendezvous studies. Various fuel-optimal rendezvous problems were formulated for the case of relative motion about an object in orbit in the equatorial plane. Similar formulations can be done for the other cases. Studies of optimal maneuvers and trajectories can therefore be performed with the equations presented herein.

As suggested by a reviewer, the basic approach of this paper extends to higher-order expansions of the gravitational fields of oblate bodies [20]. It might also be useful for first- or higher-order expansions of the gravitational fields of prolate bodies that approximate the shape of certain asteroids.

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